Nilpotency in Uncountable Groups

Marco Trombetti

Università degli Studi di Napoli Federico II

New Pathways between Group Theory and Model Theory

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Large Groups

Let $\mathcal{X}$ be a non-trivial class of groups in which every non-trivial group is infinite and let $\mathcal{U}$ be an universe of groups containing $\mathcal{X}$. Then $\mathcal{X}$ is said to be a class of large groups in the universe $\mathcal{U}$ if the following two properties are satisfied:

- If an $\mathcal{U}$-group $G$ contains a non-trivial subgroup isomorphic to a group in $\mathcal{X}$, then $G$ belongs to $\mathcal{X}$.
- If $G$ is any $\mathcal{X}$-group and $N$ is a normal subgroup of $G$, then at least one of the groups $N$ or $G/N$ belongs to $\mathcal{X}$.
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Hereinafter, by tacit agreement, we will assume that every class of groups we speak of will contain the trivial groups.
The class of **groups of infinite rank** is a class of large groups in the universe of all groups.

- Recall that a group $G$ is said to have *finite rank* $r$ if every finitely generated subgroup of $G$ can be generated by at most $r$ elements, and $r$ is the least positive integer with such a property; if such an $r$ does not exist, we will say that the group $G$ has *infinite rank*. 
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The class of **groups of uncountable cardinality** is a class of large groups in the universe of all groups.
Let $\mathcal{X}$ be a class of large groups in the universe $\mathcal{U}$.

**What happens when** $G$ is an $\mathcal{X}$-group whose all proper $\mathcal{X}$-subgroups satisfy a certain property $\mathcal{P}$?

Does the group $G$ itself satisfy the property $\mathcal{P}$?

Do all the proper subgroups of $G$ satisfy the property $\mathcal{P}$?
In some universes of (generalized) soluble groups, a group $G$ of infinite rank whose all proper subgroups of infinite rank satisfy a fixed property $\mathcal{P}$ is such that also its subgroups of finite rank have the property $\mathcal{P}$; at least for some relevant choices of the property $\mathcal{P}$.


Let $G$ be a group of uncountable cardinality $\aleph$ whose all proper subgroups of cardinality $\aleph$ satisfy a certain property $\mathcal{X}$, do all (proper) subgroups of $G$ satisfy the property $\mathcal{X}$?
The answer to this question is obstructed: Jónsson groups, namely uncountable groups with all proper subgroups of strictly smaller cardinality, blockade the road.
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The question of the existence of such groups was raised in the relevant paper of Alexander G. Kuroš and Sergeǐ N. Černikov - “Solvable and nilpotent groups” (Russian), *Uspehi Matem. Nauk (N.S.)* 2 (1947).
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In his paper “On a problem of Kurosh, Jonsson groups and applications” (1980), Saharon Shelah proved (without appeal to the continuum hypothesis) that there exists a group with cardinality $\aleph_1$ whose proper subgroups (and even subsemigroups) have cardinality strictly smaller than $\aleph_1$. 
Simplicity of Jónsson Groups

Let $G$ be a Jónsson group of cardinality $\aleph$. Then $G/Z(G)$ is a simple group of cardinality $\aleph$. 
Question

Are there uncountable groups of cardinality $\aleph$ without simple homomorphic images of cardinality $\aleph$ whose proper normal subgroups have cardinality strictly less than $\aleph$?
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An infinite cardinal $k$ is called regular if it cannot be expressed as the sum of a collection of cardinals $k_i < k$ with $i \in I$, where also the cardinality of $I$ is strictly smaller than that of $k$. 
Question

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An infinite cardinal $k$ is called **regular** if it cannot be expressed as the sum of a collection of cardinals $k_i < k$ with $i \in I$, where also the cardinality of $I$ is strictly smaller than that of $k$.

A group $G$ is called **locally graded** if every non-trivial finitely generated subgroup of $G$ has a proper subgroup of finite index.
Theorem

F. de Giovanni and M. T. – 2016

Let $\aleph$ be an uncountable regular cardinal, and let $G$ be a locally graded group of cardinality $\aleph$ which has no simple homomorphic images of cardinality $\aleph$. If all proper subgroups of $G$ of cardinality $\aleph$ are nilpotent-by-finite, then $G$ itself is nilpotent-by-finite.

F. de Giovanni, M. Martusciello and C. Rainone - 2014: Let $G$ be a group whose all proper countable subgroups are nilpotent-by-finite, then $G$ itself is nilpotent-by-finite.
Theorem

F. de Giovanni and M. T. – *Nilpotency in Uncountable Groups* (2016)

Let $G$ be an uncountable locally graded group of cardinality $\aleph$ which has no simple homomorphic images of cardinality $\aleph$. If all proper subgroups of cardinality $\aleph$ of $G$ are locally nilpotent, then $G$ itself is locally nilpotent.
Theorems

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Let $G$ be an uncountable locally graded group of cardinality $\aleph$ which has no simple homomorphic images of cardinality $\aleph$. If all proper subgroups of cardinality $\aleph$ of $G$ are locally nilpotent, then $G$ itself is locally nilpotent.

Let $G$ be a group of uncountable cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ are locally supersoluble. If the commutator subgroup $G'$ of $G$ is locally nilpotent, then $G$ is locally supersoluble.
Theorem

Let $\mathfrak{c}$ be a cardinal number whose cofinality is strictly larger than $\aleph_0$, and let $G$ be a group of cardinality $\mathfrak{c}$ which has no infinite simple homomorphic images. If all proper subgroups of $G$ of cardinality $\mathfrak{c}$ are nilpotent, then $G$ itself is nilpotent.

The condition on the cofinality of the cardinal number $\mathfrak{c}$ can be dropped out under the assumption of GCH, the generalized continuum hypothesis.
Theorem

F. de Giovanni and M. T. – *Nilpotency in Uncountable Groups* (2016)

Let $G$ be an uncountable group of cardinality $\aleph$ which has no simple non-abelian homomorphic images. If all proper subgroups of cardinality $\aleph$ are soluble with derived length at most $k$ (where $k$ is a fixed positive integer), then $G$ itself is soluble with derived length at most $k$.
Theorem

F. de Giovanni and M. T. – *Nilpotency in Uncountable Groups* (2016)

Let $G$ be an uncountable group of cardinality $\aleph$ which has no simple non-abelian homomorphic images. If all proper subgroups of cardinality $\aleph$ are soluble with derived length at most $k$ (where $k$ is a fixed positive integer), then $G$ itself is soluble with derived length at most $k$.

It seems to be an open question whether the hypothesis on the derived length can be dropped out as in the nilpotent case.
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