Projection-Invariant Subgroups

Brendan Goldsmith
Dublin Institute of Technology

Groups and Model Theory,
Mülheim, June 2nd 2011.
Recall Kaplansky’s notions of transitivity:

- A group $G$ is said to be **fully transitive** if, whenever two elements $x, y$ satisfy an obvious necessary condition for the existence of an endomorphism $\phi$ sending $x \mapsto y$, there actually is such an endomorphism.

- A group $G$ is said to be **transitive** if, whenever two elements $x, y$ satisfy an obvious necessary condition for the existence of an automorphism $\phi$ sending $x \mapsto y$, there actually is such an automorphism.

The “obvious” necessary conditions here are that the Ulm sequences obey $U_G(x) \leq U_G(y)$ and $U_G(x) = U_G(y)$ respectively.
History

An examination of Kaplansky’s LRB shows that part of his motivation for the study of these concepts came from his interest in characterizing fully invariant and characteristic subgroups. It is also clear that Kaplansky felt that all Abelian groups should be both transitive and fully transitive. This, however, was quickly shown by Megibben not to be the case. Later Corner proved that the concepts were independent

- There is a non-transitive fully transitive $p$-group $G$ with $p^{\omega}G$ an elementary group of infinite rank.
- There is a transitive non-fully-transitive 2-group $G$ with $2^{\omega}G \cong \mathbb{Z}(2) \oplus \mathbb{Z}(8)$.
- However Files + $G$ have shown that a group $G$ is fully transitive $\iff G \oplus G$ is transitive.
The Current Project

The class of fully transitive $p$-groups is very large and includes the classes of separable $p$-groups, totally projective $p$-groups and many others. It seems reasonable to focus on a slightly smaller class of groups - which one?

Considering the role played by fully invariant and characteristic subgroups in the evolution of transitivity and full transitivity, there is a natural class of subgroups to work with: the projection-invariant subgroups.

This does, indeed, lead to an interesting transitivity-type concept but our focus here will be on the projection-invariant subgroups.
Background

A subgroup $G$ of $A$ is said to be a projection-invariant subgroup if $G\pi \leq G$, for all idempotent endomorphisms $\pi$ in $\text{End}(A)$. It is straightforward to show that $G$ is projection-invariant in $A$ if, and only if, $G = (G \cap B) \oplus (G \cap C)$ for every direct decomposition $A = B \oplus C$.

Clearly every fully invariant subgroup of $A$ is a projection-invariant subgroup.
Some Known Results

• (Hausen) If $A$ is a separable $p$-group, then every projection-invariant subgroup is fully invariant.

• (Megibben) If $A$ is a transitive and fully transitive reduced $p$-group which satisfies the condition (*) below, then the projection-invariant subgroups of $A$ are fully invariant.

\[ (*) : \text{If } \alpha_1, \ldots, \alpha_n \text{ and } \beta_1, \ldots, \beta_m \text{ are two disjoint finite sequences of ordinals such that } f_A(\alpha_i) \neq 0 \text{ for each } i, \text{ then there is a direct decomposition } A = B \oplus C \text{ where } f_B(\alpha_i) = 1 \text{ for } i = 1, \ldots, n \text{ and } f_B(\beta_j) = 0 \text{ for } j = 1, \ldots, m. \]
Notions of Socle-Regularity

The notions of socle-regularity and strong socle-regularity have been introduced by Danchev and myself in an attempt to generalize the notions of transitivity and full transitivity.

Recall that a group $G$ is said to be:

- **socle-regular** if for all fully invariant subgroups $F$ of $G$, there is an ordinal $\alpha$ (depending on $F$) such that $F[p] = (p^\alpha G)[p]$;
- **strongly socle-regular** if for all characteristic subgroups $C$ of $G$, there is an ordinal $\beta$ (depending on $C$) such that $C[p] = (p^\beta G)[p]$.

Fully transitive groups are socle-regular while transitive groups are strongly socle-regular.
Projective Socle-Regularity

In light of the previous definitions, it seems reasonable to make the following definition:

• A group $G$ is said to **projectively socle-regular** if for all projection-invariant subgroups $P$ of $G$, there is an ordinal $\gamma$ (depending on $P$) such that $P[p] = (p^\gamma G)[p]$.

• Projectively socle-regular groups are socle-regular.

It is also not too surprising to discover that:

• If $p \neq 2$, then a **projectively socle-regular** group is **strongly socle-regular**.
Some Results

• If $H$ is an arbitrary finite group, then there is a projectively socle-regular group $G$ with $p^\omega G = H$.
• If $G$ is separable or totally projective, then $G$ is projectively socle-regular.
• For each prime $p$ there is a socle-regular $p$-group which is not projectively socle-regular.
• There is a strongly socle-regular 2-group which is not projectively socle-regular.
Subgroups

• If $G$ is projectively socle-regular, then so also is $p^\alpha G$ for all $\alpha$.

• If $G$ is projectively socle-regular and $L$ is a large subgroup of $G$, then $L$ is projectively socle-regular.

• If $\alpha < \omega^2$ and $p^\alpha G$ is projectively socle-regular with $G/p^\alpha G$ totally projective, then $G$ is projectively socle-regular.

• If $G/p^\beta G$ is totally projective and $p^\beta G$ is separable, then $G$ is projectively socle-regular.
Summands and Sums

• The direct sum of two projectively socle-regular groups need not be projectively socle-regular. \[ \text{Take } p^\omega A \cong \mathbb{Z}(p) \cong p^\omega B \text{ and } A/p^\omega A \text{ is } \Sigma\text{-cyclic, while } B/p^\omega B \text{ is torsion complete.} \]

• If \( G \) is projectively socle-regular and \( H \) is separable, then \( G \oplus H \) is projectively socle-regular.

On the other hand if \( A \) is projectively socle-regular and \( A = G \oplus H \), where \( H \) is separable, then it seems difficult to decide whether or not \( G \) must be projectively socle-regular.
Direct Powers

There is another source of projectively socle-regular groups which can be easily exhibited. In fact groups $G$ of the form $G = H^{(\kappa)}$, where $H$ is any $p$-group and $\kappa$ is a cardinal, have the property that every projection-invariant subgroup of $G$ is fully invariant in $G$. This follows from:

- Let $R$ be an arbitrary ring, then every $\Delta \in M_n(R)$, the ring of (finite) $n \times n$ matrices over $R$ ($n > 1$), can be expressed as a finite sum $\Delta = \Delta_1 + \cdots + \Delta_k$, where each $\Delta_i$ is either idempotent or a product of two idempotents.
Connections

If $\kappa > 1$, then the following are equivalent:

(i) $G$ is socle-regular;

(ii) $G^{(\kappa)}$ is socle-regular;

(iii) $G^{(\kappa)}$ is strongly socle-regular;

(iv) $G^{(\kappa)}$ is projectively socle-regular.

Hence a summand of a projectively socle-regular group need not be projectively socle-regular.
Problems

- Are Krylov transitive $p$-groups satisfying condition (*) projectively socle-regular groups?
- If $G$ is a socle-regular $p$-group with finite $p^\omega G$, does it follow that $G$ is projectively socle-regular?
- Is it true that projectively socle-regular 2-groups are strongly socle-regular?
References

