Envelopes and covers for groups

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(work in progress with Sergio Estrada)

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Aims

- Relate work by Enochs–Rada’05 and Hill’08, and others, on covers of abelian groups, and work by Chachólski–Farjoun–Göbel–Segev’07, Buckner-Dugas’06, Fuchs–Göbel’09, and others, on cellular covers of abelian groups.

- Extend results from module approximation theory to arbitrary groups.

- Extend results of localizations and cellular covers of groups to envelopes and covers of groups.
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1 Definitions

2 Envelopes and covers with trivial (co)-Galois groups

3 Examples of envelopes and questions
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3 Examples of envelopes and questions
Covers of groups

Let $\mathcal{F}$ be a class of groups.

A homomorphism $\pi : G \to H$ is an $\mathcal{F}$-precover if $G \in \mathcal{F}$ and for all $F \in \mathcal{F}$, it induces a surjection $\pi_* : \text{Hom}(F, G) \to \text{Hom}(F, H)$, i.e.

\[
\exists \psi \quad \forall \psi \quad \forall \psi
\]

It is an $\mathcal{F}$-cover if, furthermore, $\pi \psi = \pi$ implies $\psi$ is an automorphism of $G$. Then $\mathcal{F}$-covers are unique up to isomorphism, when they exist.
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$$F \xrightarrow{\exists \psi} \check{\forall} \psi \xrightarrow{\forall \psi} G \xrightarrow{\pi} H$$

It is an $\mathcal{F}$-cover if, furthermore, $\pi \psi = \pi$ implies $\psi$ automorphism.

coGalois group: $\text{coGal}(\pi) = \text{the group of such automorphisms}$.

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$$F \xrightarrow{\exists \psi} G \xrightarrow{\pi} H$$

$$\forall \psi$$

It is an $\mathcal{F}$-cover if, furthermore, $\pi \psi = \pi$ implies $\psi$ automorphism

$$G \xrightarrow{\psi} G \xrightarrow{\pi} H$$

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H \\
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Enochs–Rada’05, and Hill’06 considered torsion free covers of abelian groups having trivial co-Galois group. More generally:

**Theorem (1)**

Let $\mathcal{F}$ be any class of groups. Let $\pi : G \to H$ be an $\mathcal{F}$-cover. Then TFAE:

(a) $\text{coGal}(\pi) = 1$.

(b) $\pi : G \to H$ is a cellular cover, i.e. $\pi_* : \text{Hom}(G, G) \cong \text{Hom}(G, H)$. 
Trivial co-Galois groups

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- Torsion-free cellular covers have been studied by Buckner–Dugas’06, Farjoun–Göbel–Segev’07.
- Arbitrary cellular covers of groups have been studied during the last decades (Bousfield’77, Farjoun’97, R-Scherer’01, Farjoun–Göbel–Shelah–Segev’07, Göbel–R–Strüngmann’10, ...).
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Proof of Theorem 1:

(a) implies (b): Suppose \( \text{coGal}(\pi) = 1 \).

- Then \( K = \text{Ker}\,\pi \) is central in \( G \). Indeed, for \( x \in K \) consider conjugation \( c_x : G \to G \) by \( c_x(y) = xyx^{-1} \). Then, \( \pi c_x = \pi \), hence \( c_x = \text{Id}_G \).

- \( \pi_\ast \) is bijective: Let \( \psi_1, \psi_2 : G \to G \) such that \( \pi \psi_1 = \pi \psi_2 \). Define the map \( \psi : G \to G \) by

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\psi(x) = x\psi_1(x)\psi_2(x)^{-1}.
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This is a well defined homomorphism since \( \psi_1(x)\psi_2(x)^{-1} \in K \) is central. Now \( \psi\pi = \pi \), hence \( \pi = \text{Id}_G \) and we get \( \psi_1 = \psi_2 \).
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Is every envelope with trivial Galois group a localization? We do not know.

**Theorem (2)**

Let $\mathcal{F}$ be any class of groups. Let $\eta : H \to G$ be an $\mathcal{F}$-preenvelope. Assume that either $H$ abelian or $G$ nilpotent. Then TFAE:

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Let $\psi_1$ and $\psi_2$ such that $\psi_1 \eta = \psi_2 \eta$.
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Thus $\xi' = Id_G$ and $\psi_1 = \psi_2$ as desired.
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This is a homomorphism such that $\xi \eta = \eta$, hence $\xi = lG$.
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is a homomorphism such that $\xi' \eta = \eta$.
Thus $\xi' = lG$ and $\psi_1 = \psi_2$ as desired.
Localizations of groups

Localization of groups have been studied by many authors, specially during the last decade. Quite of algebraic structure is preserved (e.g. rings, modules, algebras).

Localizations of $\mathbb{Z}$ are of special interest.

- Localizations $\eta : \mathbb{Z} \to A$ are known as $E$-rings (Schultz’73). These are commutative rings $A$ with 1 with only inner additive endomorphisms.
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- Casacuberta–R–Tai’98: Homotopical localizations of \( K(\mathbb{Z}, n) \) are of the form \( K(A, n) \) where \( A \) ranges over the class of \( E \)-rings.
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- Casacuberta–Gutiérrez’05: stable homotopical localizations of $H\mathbb{Z}$ are of the form $HA$ where $A$ ranges over the class of $E$-rings.
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We are now interested on envelopes of $\mathbb{Z}$, i.e. inclusions $\eta : \mathbb{Z} \hookrightarrow G$ inducing a surjection $\eta_* : \text{Hom}(G, G) \twoheadrightarrow \text{Hom}(\mathbb{Z}, G)$, and satisfying $\psi \eta = \eta$ implies $\psi$ iso.
(Note this is an $\mathcal{F}$-envelope where $\mathcal{F} = \{ G \}$.)
These are non necessarily rings, nor abelian.

**Example**

Consider the infinite diedral group $G = \langle x, y : y^2 = 1, x^y = x^{-1} \rangle$
Then, $\langle x \rangle \hookrightarrow G$ is an envelope with Galois group $C_2$.

Plan (Göbel–R): Use Corner’s method or black boxes to construct large envelopes of $\mathbb{Z}$ with a prescribed Galois group.
Some finite envelopes

- $C_{p^r} \leftarrow C_{p^s}$ with $s > r$.
- $C_{p^r} = \mathbb{Z}/p^r \leftarrow \mathbb{Z}(p^\infty)$ (This is the injective hull).
- $C_3 \leftarrow S_3$
- $C_p \leftarrow D_{2p}$, for every prime $p$.
- $A_5 \rightarrow S_5$ is an envelope (recall $A_5 \leftarrow A_6$ is a localization).
- $G = \langle x, y : x^8 = 1, y^2 = 1, x^y = x^5 \rangle$. The cyclic group $\langle x^4 \rangle$ of order two embeds into $G$ as an envelope, although $\langle y \rangle$ does not. This cannot happen with localizations.
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Have adapted a program written by Tonks’00, designed to look for a counterexample to the following

**Open problem (Farjoun):** $H \leftrightarrow G$ localization of finite groups. $H$ nilpotent $\Rightarrow$ $G$ nilpotent?
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**Question**

When an inclusion $H \hookrightarrow G$ of finite $p$-groups is an envelope?
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Envelopes of simple groups

R–Scherer–Thevenaz’02: Considered **rigid components** of finite simple groups, as those determined by the equivalence relation given by zigzags of localizing inclusions.

Parker–Saxl’06: concluded successfully our previous work and showed that all (non-abelian) finite simple groups lie in the same rigid component, except $PSp_4(p^{2c})$, $p$ odd prime, $c > 0$, which are isolated.
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Problem
Does the weak rigid component, defined allowing zigzags of envelopes, contains these isolated groups?
To attack this problem we have the following criterium (which extends a result by R–Scherer–Thevenaz’02):

**Theorem (3)**

Let $\varphi : H \hookrightarrow G$ be a inclusion of finite simple groups. Then $\varphi$ is an envelope $\iff$

1. Every automorphism of $H$ extends to an automorphism of $G$.
2. Any subgroup of $G$ isomorphic to $H$ is conjugate to $H$ in $\text{Aut}(G)$.

*Furthermore*, $\text{Gal}(\varphi) = \text{Cen}_G(H)$.

Need another result about envelopes into alternating groups $A_n$. 
Large envelopes of simple groups

Göbel–R–Shelah'02, Göbel–Shelah'02: Every finite simple group admits arbitrarily large localizations.

Which finite simple groups admit arbitrarily large envelopes with a prescribed Galois group?
Göbel–R–Shelah’02, Göbel–Shelah’02: 

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**Problem**

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Göbel–R–Shelah’02, Göbel–Shelah'02:
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Problem
Which finite simple groups admit arbitrarily large envelopes with a prescribed Galois group?
THANK YOU!

(I include here some extra slides not presented at the talk)
If $\pi : G \to H$ is an $\mathcal{F}$-precover, and consider the epi-mono factorization

$$G \to \text{Im} (\pi) \hookrightarrow H$$

then $G \to \text{Im} (\pi)$ is also an $\mathcal{F}$-precover, with the same co-Galois group.
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Have similar result for preenvelopes.
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Hence we can study independently:

- Monomorphic (pre)covers, and surjective (pre)envelopes.
- Surjective (pre)covers, and monomorphic (pre)envelopes.
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Hence we can study independently:

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- Surjective (pre)covers, and monomorphic (pre)envelopes.
Relation with socles and radicals

Fix a group $G$ (or more generally a class of groups $\mathcal{F}$)

- The $G$-socle $S_G(H)$ of a group $H$ is defined as the subgroup generated by the images of all homomorphisms $G \to H$.

- The $G$-radical $R_G(H) := \bigcup_i R^i$ where
  - $R^0 = S_G(H)$, $R^{i+1}/R^i = S_G(H/R^i)$
  - $R^\lambda = \bigcup_\alpha R^\alpha$ if $\lambda$ is a limit ordinal.

$\rightsquigarrow$ $G$-nulification $H \mapsto H/R_GH$.

Bousfield’77, Farjoun’97:
HAVE ANALOGUE CONSTRUCTIONS IN HOMOTOPY THEORY
Relation with socles and radicals

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$\leadsto$ $G$-nullification $H \mapsto H/R_G H$.

Bousfield’77, Farjoun’97:
HAVE ANALOGUE CONSTRUCTIONS IN HOMOTOPY THEORY
More generally, fix a family $\mathcal{E}$ of epimorphisms (e.g. $\mathcal{E} = \{G \to 1\}$):

- The $\mathcal{E}$-socle of a group $H$ is the normal subgroup generated by $\psi(K)$ where $\psi : E \to H$, $K = \text{Ker}(\varphi)$ and $\varphi : E \to E'$ is in $\mathcal{E}$.

- The $\mathcal{E}$-radical is defined inductively as before.

$\mathcal{E}$-epireflection $H \to H/R_{\mathcal{E}}H$

This is localization with respect to $\mathcal{E}$. 

Proposition
Let \( \mathcal{F} \) be any class of groups. TFAE

a) \( G \to H \) is an \( \mathcal{F} \)-precover.

b) \( G \to H \) is an \( \mathcal{F} \)-cover having unique liftings.

c) \( G \) is the \( \mathcal{F} \)-socle of \( H \).

Proposition
Let \( \mathcal{F} \) be any class of groups. TFAE

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c) \( H \to G \) is the \( \mathcal{E} \)-epireflection, where \( \mathcal{E} \) is the family of epimorphisms onto \( F \in \mathcal{F} \).
Relation with socles and radicals

Proposition

Let $\mathcal{F}$ be any class of groups. TFAE

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Proposition

Let $\mathcal{F}$ be any class of groups. TFAE

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c) $H \rightarrow G$ is the $\mathcal{E}$-epireflection, where $\mathcal{E}$ is the family of epimorphisms onto $F \in \mathcal{F}$. 
Fix a group $G$. Let $C(G)$ be the class of $G$-cellular groups. This is the smallest class of groups which contains $G$ and it is closed under arbitrary colimits.

**Theorem (R-Scherer'01)**

The inclusion functor $C(G) \hookrightarrow \text{Groups}$ admits a right adjoint

$$\text{Cell}_G : \text{Groups} \to C(G).$$

This gives a $G$-cellular cover for every group $H$.

$$\pi : \text{Cell}_G H \to H$$

(i.e. $\pi_* : \text{Hom}(F, \text{Cell}_G H) \cong \text{Hom}(F, H)$, for all $G$-cellular groups $F$).
Let $M$ be a CW-complex. Bousfield’77, Farjoun’97 define $M$-cellularization for every space $X$.

**Theorem (R-Scherer’01)**

If $M = M(G, 1)$ is 2-dim Moore space then

$$\pi_1(\text{Cell}_M K(H, 1)) \simeq \text{Cell}_G H$$
Enochs’81: Existence of special $\mathcal{F}$-precovers of modules, under certain conditions.

Recall $\pi$ is a special $\mathcal{F}$-precover if $\text{Ext}(F, \ker \pi) = 0$, for all $F \in \mathcal{F}$.

We extend this to arbitrary groups $G$ “cotorsion” theories for groups. Use Quillen’s small object argument from Hirschorn’97.

**Definition**

Let $G$ be any group, and $\kappa = |G|$. Let $\aleph$ be an infinite regular cardinal $\geq \kappa^+$. Let $I$ be a set of representatives of extensions $j : N \hookrightarrow M$ by $G$, with $|M| < \aleph$.

Define the class of $G$-filtered groups as the class of $I$-cell complexes.
G-filtered groups

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Define the class of $G$-filtered groups as the class of $I$-cell complexes.
Existence of $\mathcal{F}$-covers

Suppose that $\mathcal{F}$ is the class of $I$-cell complexes, for some family $I = I_0 \cup J$ where $I_0$ is a set of monomorphisms, and $J$ is a class of epimorphisms.

**Theorem**

- Every group $H$ admits an $\mathcal{F}$-precover.
- If $J$ includes all epimorphisms $g : F \to F'$ with $F' \leq F$, and $F \in \mathcal{F}$, then each group admits an $\mathcal{F}$-cover.
- If $J$ includes the epimorphisms of the form $B \star_A B \to B$ for every inclusion $A \hookrightarrow B$ in $I_0$, then each group $H$ admits a cellular $\mathcal{F}$-cover.
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Corollary

If $F$ is closed under well ordered colimits, then every group $H$ admits an $F$-cover.

Corollary

If $F$ is closed under taking free products of coequalizers of $F$-morphisms, or equivalently closed under arbitrary colimits, then every group has a cellular $F$-cover.